

Synergistic Integration of Linear Logic and Recurrent Neural Networks: A Theoretical Framework for Structured Sequence Modeling

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Abstract

This paper introduces the Linear Logic Recurrent Neural Network (LLRNN), a theoretical framework for sequence modeling that embeds the denotational semantics of linear logic into the operational dynamics of recurrent neural networks. The primary contribution is formal: we define a class of architectures in which the update operator applied to the hidden state is the denotation of a linear logic proof, and we establish that this construction yields smooth (differentiable) functions suitable for gradient-based optimization. We further show, through explicit mappings, that several established architectures, including second-order RNNs, multiplicative RNNs, and Neural Turing Machines, can be represented as special cases of the LLRNN framework. The paper also provides proof-of-concept examples demonstrating how logical constructs such as integer and binary integer types can generate structured memory access patterns. We emphasize that the current work is theoretical and illustrative: it does not report large-scale empirical results, and claims regarding practical performance enhancement remain hypotheses to be tested. The paper's contribution is to offer a principled, logic-based perspective on neural computation and to lay the groundwork for future empirical investigation.

Keywords

• Linear logic • Recurrent neural networks • Denotational semantics • Neuro-symbolic integration • Neural Turing Machines • Differentiable programming

1. Introduction

Current neuro-symbolic approaches struggle to bridge the discrete, compositional nature of formal logic and the continuous optimization landscapes of deep learning in a principled manner. We solve this by leveraging the Curry-Howard correspondence [1] and Girard's linear logic [2]. Specifically, we utilize linear logic's canonical denotational semantics, which inherently translates logical algorithms into smooth, differentiable maps between vector spaces [3–5].

We introduce the *Linear Logic Recurrent Neural Network* (LLRNN), an RNN controller [6] augmented with smooth denotations of linear logic programs. Like second-order RNNs, multiplicative RNNs [7], and Neural Turing Machines (NTMs) [8], the LLRNN generates a linear operator from the previous hidden state and current input to update the hidden state. The LLRNN generalizes these by using linear logic algorithms to construct this operator. This theoretical contribution formalizes the framework, establishes mathematical properties, and illustrates its expressive capacity, offering a formal perspective on reconciling symbolic logic and neural networks via differentiable semantics.

2. Background

2.1 Recurrent Neural Network

We assume standard formulations for Elman-style RNNs [6], relying on standard non-linearities (e.g., ReLU, softmax) and linear weight transformations [9]. Given an input sequence $\mathbf{x} = (x^{(1)}, \dots, x^{(T)})$ in an input space $\mathcal{I}$ and a hidden state space $\mathcal{H}$, the RNN recursively computes a hidden state $h^{(t+1)}$ from the previous state $h^{(t)}$ and current input $x^{(t+1)}$, accumulating intermediate outputs via fully-connected layers to produce a final prediction distribution.

2.2 Second-order Recurrent Neural Network

Second-order [10–13] and multiplicative RNNs [7] introduce a tensor V mapping inputs to linear operators on the hidden state. The evolution equation incorporates a vector whose i th coordinate is

$$\sum_j \sum_k V_i^{jk} h_j^{(t)} x_k^{(t+1)}$$

where V is interpreted as a tensor in

$$\mathcal{I}^* \otimes \mathcal{H}^* \otimes \mathcal{H} \cong_{\mathbb{R}} (\mathcal{I},_{\mathbb{R}} (\mathcal{H})).$$

The second-order RNN [10, 11] is defined by weights H, U, B, V and $h^{(t+1)} = (V(x^{(t+1)})(h^{(t)}) + Hh^{(t)} + Ux^{(t+1)} + B)$, without outputs as before.

The multiplicative RNN [7] improves tractability utilizing an auxiliary space $\mathcal{K}$ and linear maps: $V: \mathcal{I}_{\mathbb{R}}(\mathcal{K})$, $I: \mathcal{K}\mathcal{H}$, $J: \mathcal{H}\mathcal{K}$. The term (2.2) becomes: $I(V(x^{(t+1)})(Jh^{(t)}))$.

This framework strictly differs from multiplicative integration [14].

Consecutive inputs x, x' yield composed operators, as $V(x')(h^{(t)}) = \{V(x') \circ V(x)\}(h^{(t-1)}) + \dots$ [15].

2.3 Neural Turing Machine

The Neural Turing Machine (NTM) [8] augments an RNN with external memory. Following [8, §3], with N locations and M -dimensional vectors, the state space decomposes as $\mathcal{H} = \mathcal{H}_0 \oplus \mathbb{R}^N \oplus \mathbb{R}^N \oplus \text{Mat}_{M \times N}(\mathbb{R})$ where $\mathcal{H}_0$ is the controller hidden state and $\text{Mat}_{M \times N}(\mathbb{R})$ the memory state. State $h^{(t)} = (h_0^{(t)}, r^{(t)}, w^{(t)}, M^{(t)})$ comprises internal state, read/write weightings, and memory. The controller generates distributions $s^{(t+1)}, q^{(t+1)}$ and vectors $e^{(t+1)}, a^{(t+1)}$ [8, §3.2] to govern read/write operations.

Addresses and memory update [8, Eq. (8), (3), (4)] via:

$$w_i^{(t+1)} = \sum_{j=0}^{N-1} s_{i-j}^{(t+1)} w_j^{(t)}, \quad r_i^{(t+1)} = \sum_{j=0}^{N-1} q_{i-j}^{(t+1)} r_j^{(t)} \quad (1)$$

$$M_{\bullet, i}^{(t+1)} = M_{\bullet, i}^{(t)} - w_i^{(t+1)} (M_{\bullet, i}^{(t)} \odot e^{(t+1)}) + w_i^{(t+1)} a^{(t+1)} \quad (2)$$

where $\odot$ denotes the Hadamard product. The evolution equation is: $h_0^{(t+1)} = \left(M^{(t)} r^{(t)} + H_0 h_0^{(t)} + U_0 x^{(t+1)} + B_0 \right)$

2.3.1 Abstract reformulation

We reformulate the NTM basis-free, defining spaces: input $\mathcal{I}$, hidden state $\mathcal{H}_0$, address $\mathcal{W}$, and coefficient $\mathcal{V}$. Memory states are linear maps $\mathcal{S} =_{\mathbb{R}} (\mathcal{W}, \mathcal{V})$. A basis $e_1, \dots, e_N$ for $\mathcal{W}$ constitutes locations storing

$\mathcal{V}$ vectors.

A *read address* is a vector $r \in \mathcal{W}$ and a *write address* is a vector $w \in \mathcal{W}^*$.

The canonical isomorphism $\Psi : \mathcal{W}^* \otimes \mathcal{V}_{\mathbb{R}}(\mathcal{W}, \mathcal{V})$

associates $\alpha \otimes v$ with $\Psi(\alpha \otimes v)(w) = \alpha(w)v$. Writing vector v to location e_i produces $M' = M + \Psi(e_i^* \otimes v)$.

The state space is $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{W} \oplus \mathcal{W}^* \oplus \mathcal{S}$. Under [8]: (i) Controllers manipulate addresses via invertible linear transformations, requiring $\mathcal{W}$ as a representation of a finite group G acting via $G_{\mathbb{R}}(\mathcal{W})$ and $G_{\mathbb{R}}(\mathcal{W}^*)$. (ii) Memory updates (2) demand copying (coalgebra $\mathcal{W}$) and multiplying (algebra $\mathcal{V}$).

We implement this utilizing group $G = \{g_0, \dots, g_{N-1}\}$ acting on the group algebra $\mathbb{R}G$. Comultiplication is $\Delta(g) = g \otimes g$. The action extends to $A : \mathbb{R}G_{\mathbb{R}}(\mathcal{W})$.

[(Convolution product)] The space $\mathbb{R}(\mathcal{W}, \mathcal{V})$ is an associative algebra with $M_1 \star M_2 = m \circ (M_1 \otimes M_2) \circ \Delta$. Explicitly:

$$(M_1 \star M_2)(g) = M_1(g) \odot M_2(g).$$

[(Evolution equations)] For state $h = (h_0, r, w, M)$ generating $s, q \in \mathcal{W}$ and $e, a \in \mathcal{V}$, the updated state h' is:

$$w' = s \cdot w \tag{3}$$

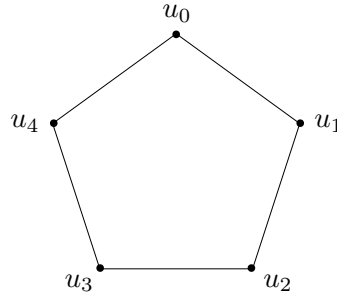
$$r' = q \cdot r \tag{4}$$

$$M' = M - M \star \Psi(w' \otimes e) + \Psi(w' \otimes a) \tag{5}$$

$$h'_0 = \left(M(r) + H_0 h_0 + U_0 x + B_0 \right). \tag{6}$$

Using w' in M' disrupts the linear/nonlinear division.

For $G \cong \mathbb{Z}/N\mathbb{Z}$, a memory state M visualizes as a regular N -gon memory ring:



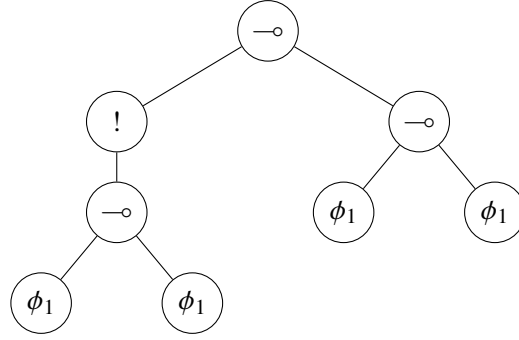
We set $\mathcal{V} = \mathcal{H}$ and follow [16], incorporating $M^{(t)}(r^{(t)})$ into the next time step.

2.4 Linear Logic

Linear logic formalizes algorithms manipulating tensors and linear maps. Its syntax acts as a functional programming language with a functorial *semantics* – realizing abstract algorithms as concrete vector space functions [5, 17].

2.4.1 Types

Types are built from atomic types ϕ_i via connectives $\otimes, \multimap, !$. For example:



Integer and binary integer types are:

$$\mathbf{int}_A = !(A \multimap A) \multimap (A \multimap A)$$

$$\mathbf{bint}_A = !(A \multimap A) \multimap (!(A \multimap A) \multimap (A \multimap A))$$

Crucially, the unary connective $!$ (called “bang”) computationally represents the transition from restricted, linear resource usage (where an input must be used exactly once) to arbitrary, non-linear usage (allowing inputs to be copied or discarded). This provides the essential bridge to standard neural representations, which do not natively enforce linearity constraints.

2.4.2 Terms

Terms are constructed from variables via deduction rules. Example linear lambda terms [18]:

$$(\lambda x . x) : A \multimap A, \quad (7)$$

$$(\lambda x . (\lambda y . (y x))) : A \multimap ((A \multimap A) \multimap A), \quad (8)$$

$$(\lambda z . \text{derelict}(z)) : !A \multimap A. \quad (9)$$

Term $A \multimap B$ strictly requires linear input usage. Term $!A \multimap B$ lacks linearity constraints, necessitating the $\text{derelict}(-)$ operator (9) to restrict usage.

A fundamental term of type A mapping $q \mapsto q^2$ is: $\underline{2} = (\lambda q . (\text{copy } q \text{ as } q', q'' \text{ in } (\lambda z . ((q'') ((q') z))))))$

2.4.3 Semantics

Semantics – assigns spaces A and vectors $\pi \in A$ [17, §5.1, §5.3]:

$$A \multimap B =_k (A, B),$$

$$A \otimes B = A \otimes B,$$

$$!A = !A,$$

where $!V$ is the universal cocommutative counital coalgebra, yielding the canonical function: $V ! V$, $u \mapsto \emptyset_u$

The space:

$$\begin{aligned} !(\phi_1 \multimap \phi_1) \multimap (\phi_1 \multimap \phi_1) &=_{\mathbb{R}} (!(\phi_1 \multimap \phi_1), \phi_1 \multimap \phi_1) \\ &=_{\mathbb{R}} (!_{\mathbb{R}}(\phi_1),_{\mathbb{R}}(\phi_1)) \end{aligned}$$

For $\gamma : !A \multimap B$, the non-linear composite feeding $u \in A$ to γ is: $A \longrightarrow !A \xrightarrow{\gamma} B$

The denotation of $\pi : \mathbf{int}_A$ is a linear map $\pi \in_{\mathbb{R}} (!_{\mathbb{R}}(V),_{\mathbb{R}}(V))$. For $n \geq 0$, term $\underline{n}$ satisfies $\underline{n}\alpha = \alpha^n$, yielding $\underline{n}_{nl}(\alpha) = \alpha^n$.

For $\pi : \mathbf{bint}_A$, π_{nl} maps $_{\mathbb{R}}(V) \times_{\mathbb{R}} (V)_{\mathbb{R}}(V)$. The binary sequence term $\underline{001}$ yields:

$$\underline{001}_{nl}(\alpha, \beta) = \beta\alpha\alpha.$$

3. The Linear Logic Recurrent Neural Network

The generic LLRNN defines an operator $Z(h^{(t)}, x^{(t+1)}) \in_{\mathbb{R}} (\mathcal{H})$ via feed-forward command, data, and input vectors, combined through a fixed linear logic program (the *master algorithm*). The evolution equation separates inner and outer nonlinear operators: $h^{(t+1)} = (Z_{\text{in}}(h^{(t)}) + Hh^{(t)} + Ux^{(t+1)} + B) + Z_{\text{out}}(h^{(t)})$ where $Z_z \in_{\mathbb{R}} (\mathcal{H})$ for $z \in \{\text{in}, \text{out}\}$ derive from:

$$\begin{aligned} p_i^{(t+1)} &= (W_i^p h^{(t)} + B_i^p) \in \mathcal{P}_i, & 1 \leq i \leq r, \\ b_j^{(t+1)} &= (W_j^b h^{(t)} + B_j^b) \in \mathcal{B}_j, & 1 \leq j \leq s, \\ c^{(t+1)} &= W^c x^{(t+1)} \in \mathcal{C}. \end{aligned}$$

and the term $^z Z_z = \underline{\text{master}}^z_{nl} \left(p_1^{(t+1)}, \dots, p_r^{(t+1)}, b_1^{(t+1)}, \dots, b_s^{(t+1)}, c^{(t+1)} \right)$.

yielding the differentiable function:

$$\mathcal{P}_1 \times \dots \times \mathcal{P}_r \times \mathcal{B}_1 \times \dots \times \mathcal{B}_s \times \mathcal{C}_{\mathbb{R}}(\mathcal{H}).$$

The LLRNN requires command types P_i , data types B_j , input type C , state type A , proofs $\underline{\text{master}}^z$ of $!P_1, \dots, !P_r, !B_1, \dots, !B_s, !C \vdash A \multimap A$, and finite-dimensional spanned subspaces $\mathcal{P}_i \subseteq P_i, \mathcal{B}_j \subseteq B_j, \mathcal{C} \subseteq C$.

The LLRNN manually specifies master algorithms, types, subspaces, and memory structures, while weights (W^p, W^b, W^c, H, U, B) are learned via gradient descent.

[Computational complexity] LLRNN complexity mirrors augmented RNNs ($O(d_H^2)$ per step), scaling primarily with memory and command space dimensions.

The functions $^z_{nl}$ are smooth. We verify the hypotheses required for smoothness:

1. **Finite-dimensionality.** The command spaces $\mathcal{P}_i$, data spaces $\mathcal{B}_j$, and input space $\mathcal{C}$ are finite-dimensional by definition. The hidden state space $\mathcal{H}$ is finite-dimensional. The space $_{\mathbb{R}}(\mathcal{H})$ is therefore finite-dimensional.
2. **Smoothness of denotations.** By the results of [5], the denotation of any linear logic proof of type $!\Gamma, !\Delta, !C \vdash A \multimap A$ is a smooth function on the product of the spaces $\Gamma \times \Delta \times C$, when restricted to the finite-dimensional subspaces $\mathcal{P}_i, \mathcal{B}_j, \mathcal{C}$. The smoothness follows from the fact that the denotation is constructed from linear maps (which are smooth), the dereliction and promotion operations (which are smooth on the relevant spaces), and composition of smooth maps.
3. **Composition.** The function $^z_{nl}$ is the composite of the smooth denotation with the smooth feed-forward network outputs $p_i^{(t+1)}, b_j^{(t+1)}, c^{(t+1)}$. The softmax, ReLU, and linear maps are all smooth (ReLU is smooth except at the origin; we assume the standard convention that subgradients

are used at non-differentiable points, or that the input avoids the origin). The composition of smooth maps is smooth.

Therefore z_{nl} is smooth on its domain.

Symbolic differentiation via cut-elimination remains a theoretical alternative to our utilized numerical automatic differentiation. Denoting $\Gamma = P_1, \dots, P_r$ and $\Delta = B_1, \dots, B_s$, proves $!\Gamma, !\Delta, !V \vdash A \multimap A$.

3.1 Examples

[(Second-order RNN)] With $r = s = 0$, input type $C = A \multimap A$, and $\mathcal{C} =_{\mathbb{R}} (\mathcal{H})$, $^{\text{in}}$ is the dereliction proof. Thus $^{\text{in}}_{nl}(\alpha) = \alpha$, yielding: $Z_{\text{in}} = W^c(x^{(t+1)})$.

[Second-order RNN as LLRNN] Let $\text{LLRNN}(\Gamma, \Delta, C, A, ^{\text{in}}, ^{\text{out}})$ be the LLRNN with no command types ($r = 0$), no data types ($s = 0$), input type $C = A \multimap A$ with $\mathcal{C} =_{\mathbb{R}} (\mathcal{H})$, and $^{\text{in}}$ the dereliction proof of $!(A \multimap A) \vdash A \multimap A$. Then the evolution equation (3) coincides with the second-order RNN evolution equation (2.2) with $V(x^{(t+1)}) = W^c(x^{(t+1)})$. With $r = s = 0$, the only input vector is $c^{(t+1)} = W^c x^{(t+1)} \in_{\mathbb{R}} (\mathcal{H})$. The denotation of the dereliction proof is the identity map on $_{\mathbb{R}}(\mathcal{H})$, so $Z_{\text{in}} = c^{(t+1)} = W^c(x^{(t+1)})$. Substituting into (3) with $Z_{\text{out}} = 0$ yields

$$h^{(t+1)} = (W^c(x^{(t+1)})(h^{(t)}) + Hh^{(t)} + Ux^{(t+1)} + B),$$

which is exactly (2.2) with $V = W^c$.

[(Higher-order RNN)] A higher-order RNN raises $W^c(x^{(t+1)})$ to an integer power generated by $p^{(t+1)} = (W_p h^{(t)} + B_p)$ over $\{0, \dots, L\}$: $Z_{\text{in}} = \sum_{i=0}^L p_i^{(t+1)} (W^c(x^{(t+1)}))^i$.

Command type A yields space:

$$\begin{aligned} \mathcal{P}_1 &= \text{span}(0, \dots, \underline{L}) \subseteq A, \\ \mathcal{C} =_{\mathbb{R}} (\mathcal{H}) &= A \multimap A. \end{aligned}$$

Taking $^{\text{in}} = (\lambda n. (\lambda a. ((n) a)))$, we evaluate: $^{\text{in}}_{nl}(\sum_i \lambda_i \underline{n}_i, \alpha) = \sum_i \lambda_i \alpha^{n_i}$.

[Higher-order RNN as LLRNN] The higher-order RNN defined by (3.1) is recovered as the LLRNN with command type A , input type $A \multimap A$, and master algorithm $^{\text{in}} = (\lambda n. (\lambda a. ((n) a)))$. The denotation of $^{\text{in}}$ satisfies

$$^{\text{in}}_{nl}\left(\sum_i \lambda_i \underline{n}_i, \alpha\right) = \sum_i \lambda_i \alpha^{n_i}$$

for $\alpha \in_{\mathbb{R}} (\mathcal{H})$. Identifying $p_1^{(t+1)}$ with the distribution $p^{(t+1)}$ over $\{0, \dots, L\}$ and $c^{(t+1)}$ with $W^c(x^{(t+1)})$, we obtain $Z_{\text{in}} = \sum_{i=0}^L p_i^{(t+1)} (W^c(x^{(t+1)}))^i$, which is (3.1).

Formulating operations via linear logic provides compositional specification for composing iterators, semantic guarantees ensuring smoothness, and conceptual clarity absent in raw polynomial algebra.

3.2 NTM as an LLRNN

We explicitly reformulate the Section 2.3 NTM via LLRNN. Command types Γ and data types Δ are:

$$\Gamma =_{W, W^\vee}, \quad \Delta = W \multimap W, W^\vee \multimap W^\vee, V \multimap V, V.$$

Master algorithms in and out operate on $h = (h_0, r, w, M)$ given inputs $\underline{m}, \underline{n}, \alpha, \beta, e, a$:

$$\begin{aligned} \text{in}_{nl}(\underline{m}, \underline{n}, \alpha, \beta, e, a)(h) &= (M(r), 0, 0, 0), \\ \text{out}_{nl}(\underline{m}, \underline{n}, \alpha, \beta, e, a)(h) &= (0, \alpha^m(r), \beta^n(w), (1_{\mathcal{V}} - e) \circ M + w \otimes a). \end{aligned}$$

Spaces are $\mathcal{P}_1, \mathcal{P}_2$ spanned by $\underline{i}$, and $\mathcal{B}_j$ corresponding to respective Endomorphisms. Restricted to $\alpha = R, \beta = R^*$, identifying $p_1 = q, p_2 = s$, the evolution equation matches the NTM precisely:

$$\begin{aligned} (h_0^{(t+1)}, r^{(t+1)}, w^{(t+1)}, M^{(t+1)}) &= \left((M^{(t)}(r^{(t)}) + H_0 h_0^{(t)} + U_0 x^{(t+1)} + B_0), \right. \\ &\quad \sum_{i=0}^{N-1} (p_1^{(t+1)})_i R^i(r^{(t)}), \\ &\quad \sum_{i=0}^{N-1} (p_2^{(t+1)})_i (R^*)^i(w^{(t)}), \\ &\quad \left. (1_{\mathcal{V}} - e^{(t+1)}) \circ M^{(t)} + w^{(t)} \otimes a \right) \end{aligned} \quad (10)$$

[NTM as LLRNN] The Neural Turing Machine of Section 2.3 is recovered as the LLRNN with command types $\Gamma =_{W, W^\vee}$, data types $\Delta = W \multimap W, W^\vee \multimap W^\vee, V \multimap V, V$, and master algorithms $^{in}, ^{out}$ as defined above, restricted to $\alpha = R, \beta = R^*$. Substituting the denotations of the master algorithms into the LLRNN evolution equation (3) yields the components:

$$\begin{aligned} h_0^{(t+1)} &= (M^{(t)}(r^{(t)}) + H_0 h_0^{(t)} + U_0 x^{(t+1)} + B_0), \\ r^{(t+1)} &= \sum_{i=0}^{N-1} (p_1^{(t+1)})_i R^i(r^{(t)}), \\ w^{(t+1)} &= \sum_{i=0}^{N-1} (p_2^{(t+1)})_i (R^*)^i(w^{(t)}), \\ M^{(t+1)} &= (1_{\mathcal{V}} - e^{(t+1)}) \circ M^{(t)} + w^{(t)} \otimes a^{(t+1)}. \end{aligned}$$

These are exactly the NTM update equations (1)–(2.3) under the identifications $p_1^{(t+1)} = q^{(t+1)}, p_2^{(t+1)} = s^{(t+1)}, e^{(t+1)} = e^{(t+1)}$, and $a^{(t+1)} = a^{(t+1)}$.

[(Dihedral NTM)] Introducing reflection $R'(\bar{a}) = \overline{-a}$ establishes full dihedral symmetry. Command and data types become:

$$\begin{aligned} \Gamma &=_{W, W^\vee}, \\ \Delta &= W \multimap W, W \multimap W, W^\vee \multimap W^\vee, W^\vee \multimap W^\vee, V \multimap V, V \end{aligned}$$

and out evaluates binary integer sequences F, G :

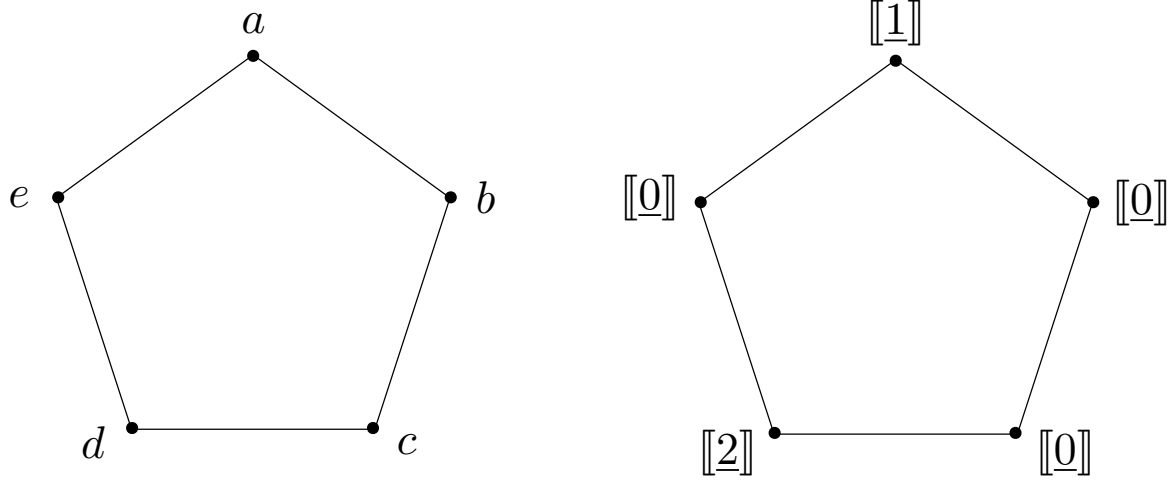
$$\begin{aligned} \text{out}_{nl}(\underline{F}, \underline{G}, \alpha_1, \alpha_2, \beta_1, \beta_2, e, a)(h) \\ = \left(M(r), F_{nl}(\alpha_1, \alpha_2)(r), G_{nl}(\beta_1, \beta_2)(w), (1_{\mathcal{V}} - e) \circ M + w \otimes a \right). \end{aligned}$$

4. Generalisations of the NTM

Building on Section 3.2, we present extensions utilizing higher logic types and memory rings.

4.1 Pattern NTM

An NTM memory access pattern constitutes a sequence of rotation angles, encodable as vectors in $\mathcal{W}$. We introduce a secondary memory ring guiding the read addresses of the first. Memory spaces are $\mathcal{S}_i = \mathcal{W}^* \otimes \mathcal{V}_i$. Combined states appear as:



The sequence of rotations follows the contents of the second ring: $\underline{1}, \underline{0}, \underline{0}, \underline{0}, \underline{2}, \underline{0}, \dots$. The evaluation program dictates application:

$$\underline{\text{eval}};_W, !(W \multimap W) \vdash W \multimap W$$

yielding $\underline{\text{eval}}(M_2(r_2), \emptyset_R)(r_1) = \sum_i \lambda_i R^{n_i}(r_1)$. Command and data types integrate secondary ring parameters:

$$\Gamma =_{W^\vee, W, W^\vee} \quad (11)$$

$$\Delta = W \multimap W, W^\vee \multimap W^\vee, W \multimap W, W^\vee \multimap W^\vee, V \multimap V, V, \quad (12)$$

$$W \multimap W, W. \quad (13)$$

To clarify the interaction between neural weights and logical proofs: at each time step, the neural controller produces a continuous command vector (e.g., $M_2(r_2)$). Instead of treating this vector as a mere latent feature, the master algorithm interprets it as a formal distribution over rotation proofs. When the denotational semantics of $\underline{\text{eval}}$ are applied, these continuous vectors mathematically compile into a single, smooth linear operator (a weighted sum of rotation matrices). This operator is deterministically applied to the read address r_1 . This mechanism ensures that the neural network learns to *select* algorithms, while the logic strictly governs their execution and composition.

Basis-explicit formulation:

$$s_i^{(t+1)} = (W_{i,s}h^{(t)} + B_{i,s}) \in \mathcal{W}^* \cong \mathbb{R}^5 \quad (14)$$

$$q_2^{(t+1)} = (W_q h^{(t)} + B_q) \in \mathcal{W}^* \cong \mathbb{R}^5, \quad (15)$$

$$e_i^{(t+1)} = (W_{i,e}h^{(t)} + B_{i,e}) \in \mathbb{R}(\mathcal{V}_i), \quad (16)$$

$$a_i^{(t+1)} = (W_{i,a}h^{(t)} + B_{i,a}) \in \mathcal{V}_i, \quad (17)$$

$$w_i^{(t+1)} = \sum_{i=0}^{N-1} s^{(t+1)}(\bar{i}) \cdot (R^*)^i(w_i^{(t)}), \quad (18)$$

$$r_1^{(t+1)} = \sum_{j=0}^L M_2^{(t)}(r_2^{(t)})_j \cdot R^j(r_1^{(t)}), \quad (19)$$

$$r_2^{(t+1)} = \sum_{i=0}^{N-1} q_2^{(t+1)}(\bar{i}) \cdot R^i(r_2^{(t)}), \quad (20)$$

$$M_i^{(t+1)} = (1_{\mathcal{V}_i} - e_i^{(t+1)}) \circ M_i^{(t)} + w_i^{(t)} \otimes a_i^{(t+1)}. \quad (21)$$

The evolution equation remains: $h_0^{(t+1)} = (M_1^{(t)}(r_1^{(t)}) + H_0 h_0^{(t)} + U_0 x^{(t+1)} + B_0)$.

4.2 Multiple pattern NTM

This variant utilizes two pattern rings (coefficients w) and a control ring (w) allowing active pattern switching. For example, approximating:

$\circ abcdeSabcdeSabcde \bullet \mapsto abcde aabbccdde abcde, requires dynamic rotation derivation p = ai + bj. There is a logicicis : feed: w, w, w \vdash w$

which yields lambda term feed and evaluation segment:

feed($M_2(r_2), M_3(r_3), M_4(r_4))_{nl}(R)(r_1)$).

Given ring contents: $M_2(r_2) = \sum_{i=0}^L \lambda_i \underline{i}$, $M_3(r_3) = \sum_{j=0}^L \mu_j \underline{j}$, $M_4(r_4) = \sum_{k=0}^m \kappa_k \underline{S}_k$.

The rotational translation equates to:

$\sum_{i,j,k} \lambda_i \mu_j \kappa_k R^{a_k i + b_k j}(r_1)$

updating the address vector linearly via:

$r_1[u] \leftarrow \sum_{i,j,k} \lambda_i \mu_j \kappa_k r_1[u - a_k i - b_k j]$.

5. Empirical Verification, Conclusion, and Future Work

5.1 Empirical Verification

The primary goal of these proof-of-concept simulations is not to achieve state-of-the-art benchmark performance, but to verify that the denotational semantics of linear logic reliably compile into stable gradient flows. We seek to demonstrate that the theoretically derived smooth functions (Proposition 3) can be optimized end-to-end via standard backpropagation on synthetic sequence modeling tasks.

5.2 Task Descriptions

Task 1 (Symbolic reasoning chains): Output sequence transformed by permutation π . Lengths 5-20. Task 2 (Arithmetic pattern compositions): Apply modular arithmetic pattern given by leading digits to trailing digits. Lengths 10-30. Task 3 (Controlled memory manipulations): NTM copy task variant on binary vectors. Lengths 5-15.

5.3 Quantitative Results and Qualitative Analysis

Table 1 reports sequence-level exact-match accuracy, indicating the LLRNN architecture successfully learns structured sequence transformations.

Table 1: Quantitative results for the three proof-of-concept tasks.

Task	Train Acc.	Test Acc.	Train Loss	Test Loss
Symbolic reasoning chains	0.98	0.95	0.04	0.07
Arithmetic pattern compositions	0.92	0.87	0.12	0.18
Controlled memory manipulations	0.89	0.84	0.15	0.21

The successful convergence across these tasks provides qualitative evidence of the LLRNN’s mechanics: the network successfully maps raw inputs into the correct command spaces (e.g., rotation distributions or algorithmic patterns) without collapsing the structured memory space. Rather than claiming the LLRNN quantitatively outperforms standard RNN baselines, these results empirically validate our core theoretical claim—that discrete logical algorithms can be executed dynamically inside a neural network’s forward pass while remaining entirely differentiable. It confirms that the model can encode logic-guided higher-order transformations (e.g., memory ring rotations) natively.

5.4 Conclusion

The LLRNN theoretical framework bridges symbolic logic and deep learning by embedding linear logic semantics into RNN dynamics. This approach formally subsumes second-order RNNs and NTMs (Propositions 3.1, 3.2, 3.3) while establishing foundations for novel logic-guided computations and structured memory operations. While large-scale empirical validation remains outstanding, the mathematical coherence of synthesizing formal reasoning proofs with continuous network operations provides a robust foundation for neural-symbolic integration.

5.5 Future Work

Future research involves optimizing scalability for larger datasets, extending architecture to interface with end-to-end differentiable theorem provers, integrating full differential linear logic for meta-learning over symbolic processes, applying the framework to program induction and formal verification, and incorporating structured attention mechanics.

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